whenever $n > n_0$. Therefore,

$$\left|\int_{x_n}^{y_n} f(t)dt - f(c)(y_n - x_n)\right| \le \int_{x_n}^{y_n} |f(t) - f(c)|\, dt \le \epsilon\, |y_n - x_n|\,.$$

Note that the given integral equals

$$I_n = n\int_{\frac{n}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{n\sqrt[n+1]{(n+1)!}}} f(t)dt,$$

this comes directly from the substitution $t = \dfrac{x}{n}$. Let $x_n, y_n$ be the lower, upper bound of the last integral respectively then $x_n, y_n \to e$, since $\dfrac{n}{\sqrt[n]{n!}} \to e$, and thus $\dfrac{(n+1)^2}{n\,\sqrt[n+1]{(n+1)!}} = \dfrac{n+1}{n}\dfrac{(n+1)}{\sqrt[n+1]{(n+1)!}} \longrightarrow e$, as $n \longrightarrow \infty$. Note that by Stolz' theorem

$$n(y_n - x_n) = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \longrightarrow e, \text{ as } n \longrightarrow \infty.$$

By the lemma we have

$$I_n = n\,[f(c)(y_n - x_n) + O(y_n - x_n)] = ef(e) + O(1),$$

which proves that the limit equals $ef(e)$.

**Also solved by Arkady Alt, San Jose, CA; Kee-Wai Lau, Hong Kong, China; Soumitra Mandal, Scottish Church College, Chandan -Nagar, West Bengal, India; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova, Varna, Bulgaria, and the proposers.**

**5467:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

In an arbitrary triangle $\triangle ABC$, let $a, b, c$ denote the lengths of the sides, R its circumradius, and let $h_a, h_b, h_c$ respectively, denote the lengths of the corresponding altitudes. Prove the inequality

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \ge \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}},$$

and give the conditions under which equality holds.

**Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy**

We know that $h_a = (bc)/(2R)$ and cyclic so the inequality actually is

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \ge \frac{3abc}{2R}\left(\frac{8R^3}{(abc)^2}\right)^{\frac{1}{3}} = 3(abc)^{\frac{1}{3}}.$$

We prove the stronger one

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c,$$

that is

$$\left(\frac{a^2+bc}{b+c}-a\right)+\left(\frac{b^2+ca}{c+a}-b\right)+\left(\frac{c^2+ab}{a+b}-c\right)\geq 0,$$

or

$$\frac{(a-b)(a-c)}{b+c}+\frac{(b-c)(b-a)}{a+c}+\frac{(c-a)(c-b)}{a+b}\geq 0.$$

We can suppose $a\geq b\geq c$ by symmetry so we come to

$$\frac{(a-b)(a-c)}{b+c}+\frac{(a-c)(b-c)}{a+b}\geq\frac{(a-b)(b-c)}{a+c}.$$

This is implied by

$$\frac{(a-b)(a-c)}{b+c}+\underbrace{\frac{(a-b)(b-c)}{a+b}}_{a-c\geq a-b}\geq\frac{(a-b)(b-c)}{a+c},$$

or

$$\frac{a-c}{b+c}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}.$$

This is in turn implied by

$$\underbrace{\frac{a-c}{a+c}}_{a\geq b}+\frac{b-c}{a+b}\geq\frac{b-c}{a+c}$$

and this evidently holds true by $a-c\geq b-c\geq 0$. The equality case is $a=b=c$.

**Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX**

We will prove the following slight improvement:

$$\begin{aligned}\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}&\geq a+b+c\\&\geq 3\sqrt[3]{abc}\\&=\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a\cdot h_b\cdot h_c}},\end{aligned}\tag{1}$$

with equality if and only if $a=b=c$.

To begin, we note that

$$\begin{aligned}a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2&=\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{2}\\&\geq 0,\end{aligned}\tag{2}$$

with equality if and only if $a^2 = b^2 = c^2$. Since $a, b, c > 0$, it follows that equality is attained in (2) if and only if $a = b = c$.

Next, we use (2) to obtain

$$
\begin{aligned}
&\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \\
&= \frac{\left(a^2-b^2\right)+\left(b^2+bc\right)}{b+c} + \frac{\left(b^2-c^2\right)+\left(c^2+ca\right)}{c+a} + \frac{\left(c^2-a^2\right)+\left(a^2+ab\right)}{a+b} \\
&= \frac{a^2-b^2}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \frac{\left(a^2-c^2\right)+\left(c^2-b^2\right)}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} + a+b+c \\
&= \left(a^2-c^2\right)\left(\frac{1}{b+c} - \frac{1}{a+b}\right) + \left(b^2-c^2\right)\left(\frac{1}{c+a} - \frac{1}{b+c}\right) + a+b+c \\
&= \left(a^2-c^2\right)\frac{a-c}{(a+b)(b+c)} + \left(b^2-c^2\right)\frac{b-a}{(b+c)(c+a)} + a+b+c \\
&= \frac{\left(a^2-c^2\right)^2 + \left(b^2-c^2\right)\left(b^2-a^2\right)}{(a+b)(b+c)(c+a)} + a+b+c \\
&= \frac{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2}{(a+b)(b+c)(c+a)} + a+b+c \\
&\geq a+b+c, \qquad (3)
\end{aligned}
$$

with equality if and only if $a = b = c$.

Also, the Arithmetic - Geometric Mean Inequality implies that

$$a+b+c \geq 3\sqrt[3]{abc}, \qquad (4)$$

with equality if and only if $a = b = c$.

For the final step, let $K = area\,(\triangle ABC)$. Then,

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$$

and hence,

$$h_a = \frac{2K}{a}, \quad h_b = \frac{2K}{b}, \quad \text{and} \quad h_c = \frac{2K}{c}.$$

Since $R = \dfrac{abc}{4K}$, we have

$$
\begin{aligned}
\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a \cdot h_b \cdot h_c}} &= \frac{12KR}{2R}\sqrt[3]{\frac{abc}{8K^3}} \\
&= \frac{6K}{2K}\sqrt[3]{abc} \\
&= 3\sqrt[3]{abc}. \qquad (5)
\end{aligned}
$$

If we combine (3), (4), and (5), statement (1) follows and equality is attained throughout if and only if $a = b = c$.

**Solution 3 by Arkady Alt, San Jose, CA**

Let $F = [ABC]$ (area) and let $s$ be its semi-perimeter.
Since $h_a = \frac{2F}{a}, h_b = \frac{2F}{b}, h_c = \frac{2F}{c}$ and $abc = 4RF$ then

$$\sqrt[3]{\frac{1}{h_a h_b h_c}} = \sqrt[3]{\frac{abc}{8F^3}} = \frac{1}{2F}\sqrt[3]{abc} \text{ and}$$

$$\frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}} = 3\sqrt[3]{abc}.$$

Thus, original inequality becomes

$$\textbf{(1)} \qquad \frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq 3\sqrt[3]{abc}.$$

Since $\frac{4a^2}{b+c} \geq 4a - b - c \iff (2a-b-c)^2 \geq 0$ we have

$$\sum_{cyc} \frac{a^2+bc}{b+c} = \sum_{cyc} \frac{a^2}{b+c} + \sum_{cyc} \frac{bc}{b+c} \geq \sum_{cyc} \frac{4a-b-c}{4} + \sum_{cyc} \frac{bc}{b+c}$$

$$= \frac{a+b+c}{2} + \sum_{cyc} \frac{bc}{b+c} = \sum_{cyc} \left(\frac{b+c}{4} + \frac{bc}{b+c}\right) \geq \sum_{cyc} 2\sqrt{\frac{b+c}{4} \cdot \frac{bc}{b+c}}$$

$$= \sum_{cyc} \sqrt{bc} \geq 3\sqrt[3]{\sqrt{bc} \cdot \sqrt{ca} \cdot \sqrt{ab}} = 3\sqrt[3]{abc}.$$

**Solution 4 by Nicusor Zlota,"Traian Vuia" Technical College, Focsani, Romania, and Corneliu-Manescu Avram, Ploiesti, Romania**

Assume that $a \geq b \geq c$.

First, we will prove that $\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c \iff$

$$\frac{a^2+bc}{b+c} - a + \frac{b^2+ca}{c+a} - b + \frac{c^2+ab}{a+b} - c \geq 0 \iff$$

$$\frac{(a-b)(a-c)}{b+c} + \frac{(b-c)(b-a)}{c+a} + \frac{(c-a)(c-b)}{a+b} \geq 0 \iff$$

$$(a-b)\left(\frac{a-c}{b+c} - \frac{b-c}{c+a}\right) + (b-a)\left(\frac{b-a}{c+a} - \frac{c-a}{a+b}\right) + (c-a)\left(\frac{a-c}{b+c} - \frac{b-c}{c+a}\right) \geq 0$$

$$(a-b)^2\frac{a+b}{(b+c)(c+a)} + (b-c)^2\frac{b+c}{(a+b)(c+a)} + (c-a)^2\frac{c+a}{(a+b)(b+c)} \geq 0.$$

Then, it suffices to prove that

$$a=b=-c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}=\frac{3abc}{2R}\sqrt[3]{\frac{abc}{8S^3}}=\frac{3abc}{2R}\frac{1}{2S}\sqrt[3]{abc}=\sqrt[3]{abc},$$

which is the AM-GM inequality.

Equality holds for $a=b=c$.

**Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy Rehovot, Israel; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro,GA, and the proposer.**

*Editor's note*: **Hatef I. Arshagi's solution was dedicated to the memory of Mrs. Alieh Ataee**.

**5468:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Find all differentiable functions $f:\Re\to\Re$ with $f(0)=1$ such that $f'(x)=f^2(-x)f(x)$, for all $x\in\Re$.

**Solution 1 by Moti Levy, Rehovot, Israel**

Let us differentiate both sides of the given differential equation,

$$f^{''}(x)=-2f(-x)f^{'}(-x)\,f(x)+f^2(-x)f^{'}(x). \tag{1}$$

The following two equation are direct consequence of the original equation.

$$f^{'}(-x)=f^2(x)f(-x), \tag{2}$$

$$f^2(-x)=\frac{f^{'}(x)}{f(x)}. \tag{3}$$

After substitution of (2) and (3) in (1), we get differential equation (4) with initial conditions at $x=0$,

$$f^{''}(x)+2f^2(x)f^{'}(x)-\frac{\left(f^{'}(x)\right)^2}{f(x)}=0,\quad f\,(0)=f^{'}\,(0)=1. \tag{4}$$

By the substitution $f\,(x)=\sqrt{g\,(x)}$,

$$\begin{aligned} f &= \sqrt{g},\\ f^{'} &= \frac{1}{2\sqrt{g}}g^{'},\\ f^{''} &= \frac{1}{2\sqrt{g}}g^{''}-\frac{1}{4\left(\sqrt{g}\right)^3}\left(g^{'}\right)^2, \end{aligned}$$